Self-similarity and multifractality in human brain - a wavelet analysis of MEG scale-free dynamics

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Outline

Scale free dynamics in Brain

Modeling

Analysis

Multifractal

Data

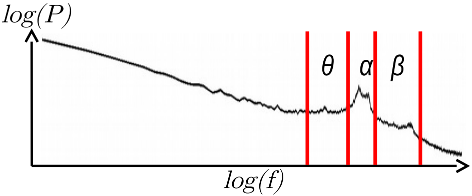
Scale-free dynamics

Conclusions

Scale-Free dynamics in Brain Activity

- Brain activity description:

  Oscillation versus Scale-free ?

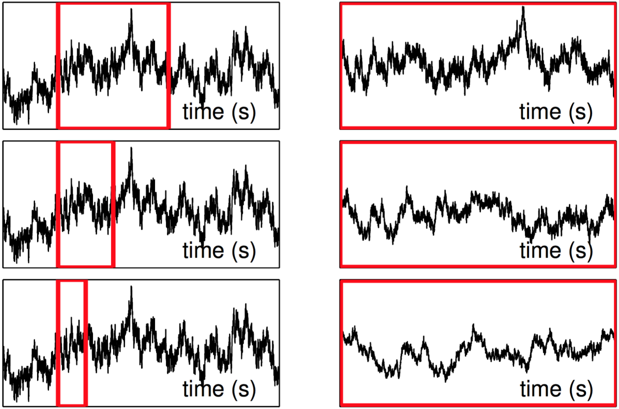


- Scale-free: Controversial !
  - \- Instrumental noise ? Movements ?
  - \+ Infraslow activity (below 1 Hz), Large energy consumption
  - \+ At rest and during task
  - \+ Modulated by task engagement, by pathologies

$\Longrightarrow$ Important to study

Outline

Scale-free dynamics: Intuition

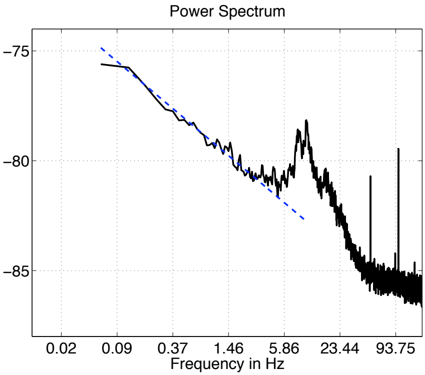


- Covariance under Dilation (Change of Scale),
- The Whole and the SubPart (Statistically) Undistinguishable,
- No Characteristic Scale of Time

$1/f$-process (Model 1)

- 2nd order stationary $1/f$-process

$$\Gamma_Y(\nu) = C|\nu|^{-\gamma}, \ \gamma > 0, \ \ \nu_m \leq |\nu| \leq \nu_M, \ \frac{\nu_M}{\nu_m} \gg 1$$



Data MEG, Courtesy, Ph. Ciuciu, Neurospin, France

Self-Similar process (Model 2)

- Definition: $\{X(t)\}_{t\in\mathcal{R}} \stackrel{fdd}{=} \{a^H X(t/a)\}_{t\in\mathcal{R}}$
  Dilation Factor: $\forall a > 0$,
  Self-Similarity Exponent: $H > 0$.
- Scaling:

Selfaffine

Self-Similar process and stationary increments

- Stationary increments:
  $\{X(t+\tau) - X(t)\}_{t\in\mathcal{R}} \stackrel{fdd}{=} \{X(0+\tau) - X(0)\}_{t\in\mathcal{R}}$

- Finite variance: $\mathbf{E}X(t)^2 < +\infty$

$\Rightarrow$ $0 < H < 1$,

$\Rightarrow$ $\mathbf{E}X(t)X(s) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \sigma^2 = \mathbf{E}X(1)^2$

$\Rightarrow$ Stability under addition (Gaussian, α-stable, Hermite)

$\Rightarrow$ Power law spectrum for increments with $\gamma = 2H - 1$.

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Scale Invariance: Spectrum Analysis (Analysis Tool 1)

- Spectrum Estimation

  (time windowed) Periodogram or Welch estimator
  $|\tilde{Y}_{k,T}(\nu)|^2$ spectral estimate in $t \in [t_k - T/2, t_k + T/2]$
  $\hat{\Gamma}_Y(\nu) = \sum_k |\tilde{Y}_{k,T}(\nu)|^2$

- $1/f$-spectrum:

  $\frac{1}{K} \sum_k |\tilde{Y}_{k,T}(\nu)|^2 = C|\nu|^{-\gamma}$

- Estimation:

  $\hat{\gamma} \rightarrow \log \frac{1}{K} \sum_k |\tilde{Y}_{k,T}(\nu)|^2$ versus $\log |\nu|$
  involve $\hat{\gamma}$ in analysis, detection, classification

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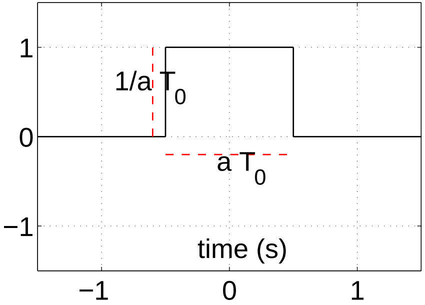
Scaling analysis: Aggregation

Average within box of size a

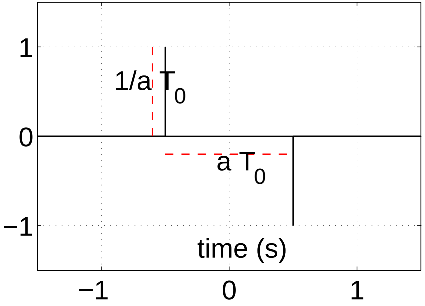
$$T_X(a,t) = \frac{1}{aT_0}\int_t^{t+aT_0} X(u)du$$



Scaling analysis: Increments

Difference over step lag of size a

$$T_X(a, t) = X(t + aT_0) - X(t)$$



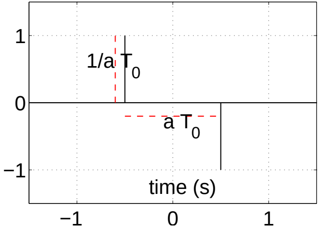
Scaling analysis: multiresolution analysis

- $X(t) \to T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0(\frac{u-t}{a})$

increment

difference

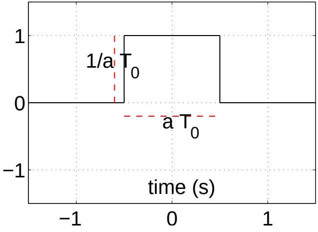
$X(t + aT_0) - X(t)$



Aggregation

average

$\frac{1}{aT_0} \int_t^{t+aT_0} X(u)du$



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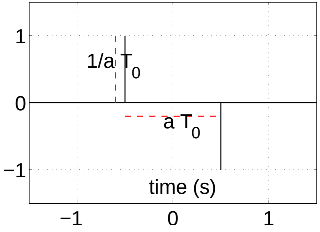
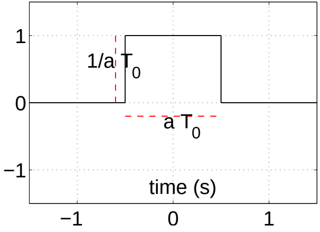
increment	Aggregation	?
difference	average	?
$X(t + aT_0) - X(t)$	$\frac{1}{aT_0}\int_t^{t+aT_0} X(u)du$	?





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Multiresolution analysis

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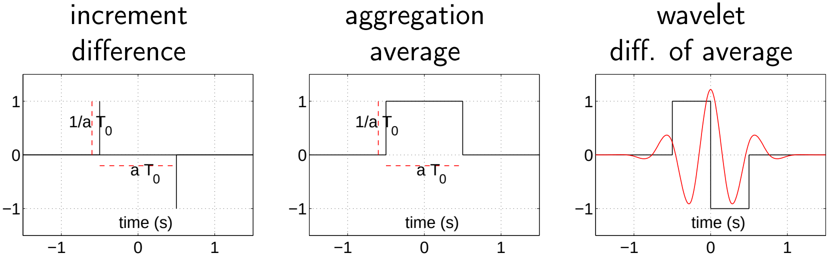

Continuous Wavelet Transform

- Fourier Transform: $X(t) \Longrightarrow \tilde{X}(\nu) = \langle X, e_\nu \rangle$.
  Fourier Basis: $e_\nu(t) = \exp(\imath 2\pi\nu t)$
  Interpretation: ever lasting pure tone

- Continuous Wavelet Transform: $T_X(a, t) = \langle X, \psi_{a,t} \rangle$

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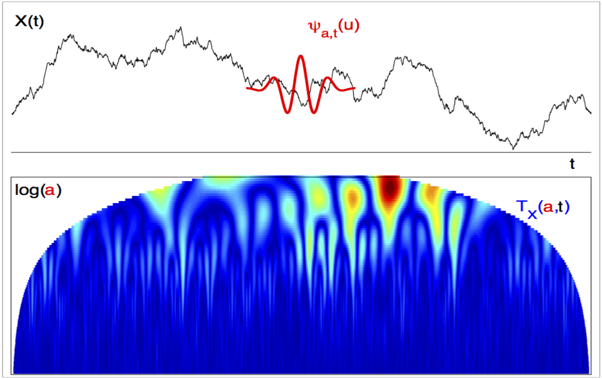

Multiresolution analysis (Analysis tool 2)

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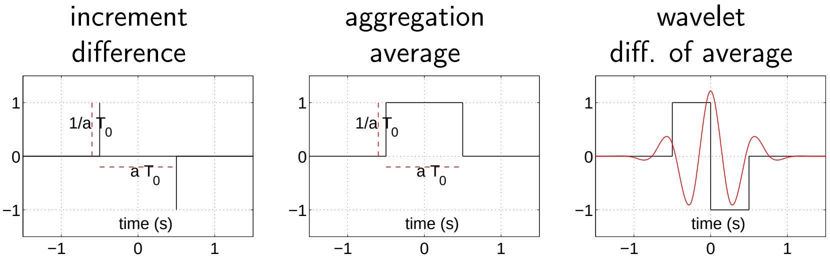


N_ψ
Number of vanishing moments

Scaling analysis: Logscale Diagrams

- Principle:

$$\mathbf{E}|d_X(j,k)|^q = |d_X(0,0)|^q 2^{jqH} \Rightarrow \text{log-log plots}$$

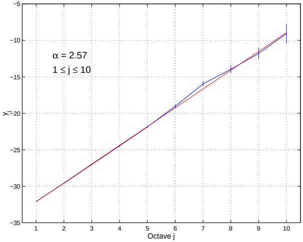
- Estimation: short-range dependence $\Rightarrow$

Ensemble averages $\rightarrow$ Time Averages

$$\mathbf{E}|d_X(j,k)|^q \Rightarrow 1/n_j \sum_k |d_X(j,k)|^q = S(2^j, q)$$

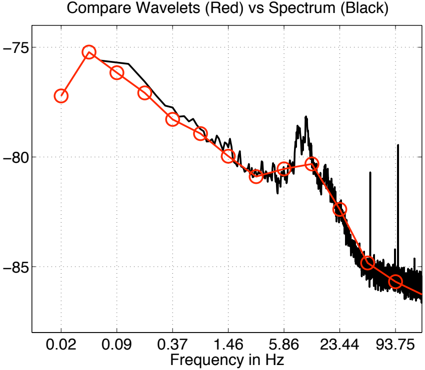
- Logscale Diagrams:

$$\log_2 S(2^j, q) \text{ versus } \log_2 2^j = j \Rightarrow qH$$



Spectrum Analysis versus Wavelet Analysis

- Wavelet Analysis: $\mathbf{E}|T_X(a,k)|^2 = \int \Gamma_X(\nu)|\tilde{\Psi}(a\nu)|^2 d\nu$
- $1/f-$process: $\frac{1}{n_a}\sum_{k=1}^{n_a}|T_X(a,k)|^2 \simeq C_q a^{\gamma-1}$
- $1/f-$process: $\hat{\Gamma}_Y(\nu) = \sum_k |\tilde{Y}_{k,T}(\nu)|^2 \simeq C|\nu|^{-\gamma}$
- $\nu \simeq \nu_0/a$ and $q = 2$



Data FMRI, Courtesy, Ph. Ciuciu, Neurospin, France

Outline

Beyond self-similarity. . . ?

- Self-Similarity:

  Power Laws: $\mathbb{E}|d_X(j,k)|^q = C_q(2)^{jqH}$
  For all scales: $\forall a = 2^j$,
  For all orders: $q > -1$,
  A single parameter qH.

- Beyond:

  Power Laws: $\mathbb{E}|d_X(j,k)|^q = C_q(2)^{j\zeta(q)}$
  $\zeta(q)$ non linear concave function of q,
  For a limited range of scales: $a_m \leq a \leq a_M$,
  For a limited range of orders: $q_m \leq q \leq q_M$,
  A collection of scaling parameters $\zeta(q)$.

$\Rightarrow$ Multifractal

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$\Rightarrow$ Multifractal

What does multifractality look like ?

What does multifractality sound like ?

fBm ($H = 0.25$)

fBm ($H = 0.75$)

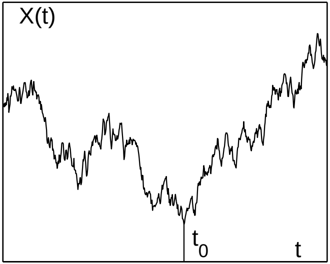
mrw ($c_1 = 0.75, c_2 = -0.03$)

mrw ($c_1 = 0.75, c_2 = -0.06$)

Multifractal Analysis

- **Local regularity** of $X(t)$ at t_0 : $0 < \alpha < 1$

  Compare: $|X(t) - X(t_0)| < C|t - t_0|^{\alpha}$



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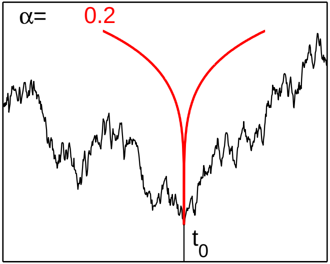

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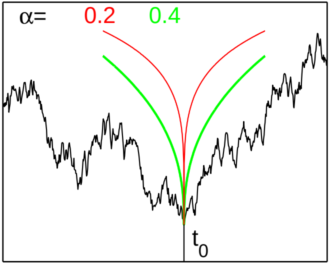

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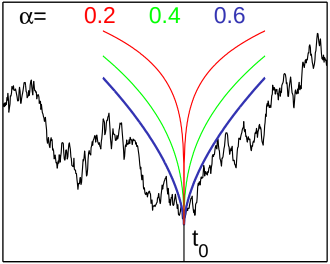

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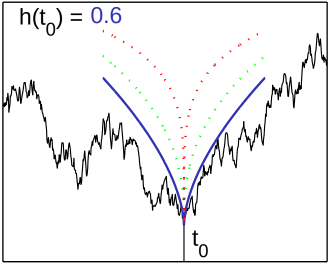
- **Hölder Exponent** : $h(t_0) = \sup_{\alpha}\{\alpha : X \in C^{\alpha}(t_0)\}$

  Extend differentiability to non integer : $0 < h(t_0) < 1$

  $\lim_{|\mathbf{t}-\mathbf{t_0}|\to 0} \frac{|X(\mathbf{t})-X(\mathbf{t_0})|}{|\mathbf{t}-\mathbf{t_0}|^{h(t_0)}} = C$



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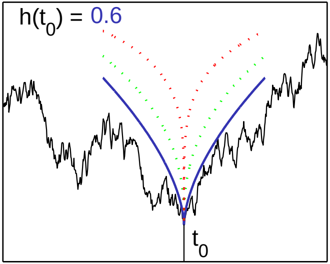
  $h(t_0) \to 1 \Rightarrow$, smooth, very regular,

  $h(t_0) \to 0 \Rightarrow$, rough, very irregular



Multifractal (or singularity) spectrum

- Data: a collection of singularities
  $|X(\mathbf{t}) - X(\mathbf{t_0})| \leq C|\mathbf{t} - \mathbf{t_0}|^{h(\mathbf{t_0})}$
- Fluctuations of local regularity: $h(\mathbf{t})$?
- not interested in h for each $\mathbf{t}$!
- Instead, set $E(h)$ of points $\mathbf{t}$ with same h: $h(\mathbf{t}) = h$,
- Fractal dimension of $E(h)$,
- Actually Hausdorff dimension of $E(h)$, ▸ Hausdorff
- Multifractal spectrum: ▸ $D(h)$

$$D(h) = \dim_{\text{Haussdorf}}(E(h)).$$

$$0 \leq D(h) \leq d,$$
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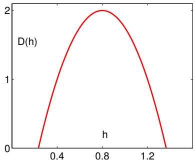

$\Rightarrow$ Global/geometrical description of local regularity fluctuations

- How to measure $D(h)$ from a single finite length observation?

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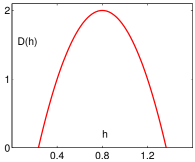

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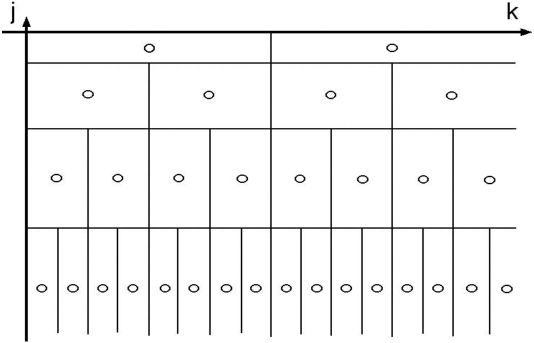
Wavelet Leaders

- Discrete Wavelet Transform: $\lambda_{j,k} = [k2^j, (k+1)2^j)$

$$d_X(j,k) = \langle \frac{1}{2^j} \psi \left(\frac{t - 2^j k}{2^j} \right) | X(t) \rangle ,$$



- Wavelet Leaders: $3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$

$$L_X(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d_{X,\lambda'}|$$

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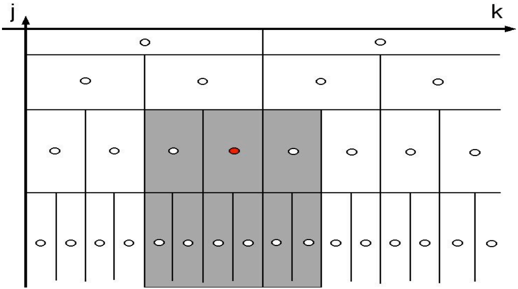
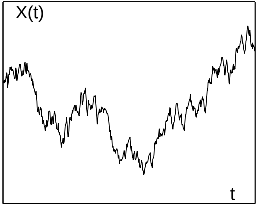


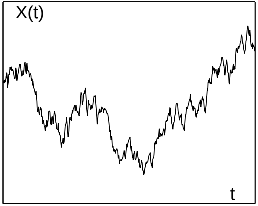
Multifractal Formalism (Analysis tool 3)

$X(t) \rightarrow d_X(a, t) \rightarrow L_X(a, t)$



Multifractal Formalism (Analysis tool 3)

$X(t) \rightarrow d_X(a, t) \rightarrow L_X(a, t)$

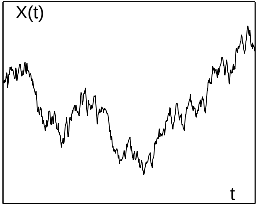


$\Downarrow$

$S(a, q) = \frac{1}{n_a} \sum_{k=1}^{n_a} L_X(a, k)^q$

Multifractal Formalism (Analysis tool 3)

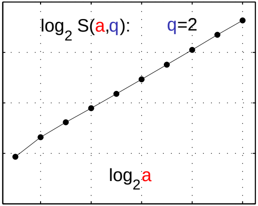
$X(t) \to d_X(a,t) \to L_X(a,t)$



⇓

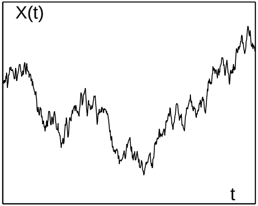
$S(a,q) = \frac{1}{n_a}\sum_{k=1}^{n_a} |L_X(a,k)|^q$

$S(a,q) \simeq c_q a^{\zeta(q)}, \quad a \to 0$



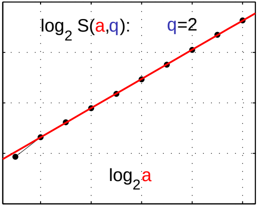
Multifractal Formalism (Analysis tool 3)

$X(t) \to d_X(a,t) \to L_X(a,t)$



$\Downarrow$

$S(a,q) = \frac{1}{n_a}\sum_{k=1}^{n_a} |L_X(a,k)|^q$



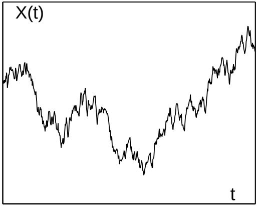
$S(a,q) \simeq c_q a^{\zeta(q)}, \quad a \to 0$

$\zeta(q) = \liminf_{a\to 0} \frac{\ln S(a,q)}{\ln a}$

Multifractal Formalism (Analysis tool 3)

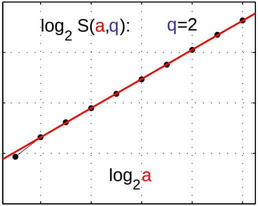
$X(t) \to d_X(a,t) \to L_X(a,t)$

$D(h) = \min_{q \neq 0} (d + qh - \zeta(q))$



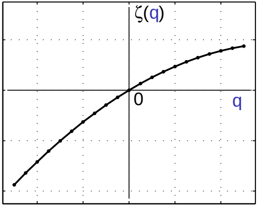
$\Downarrow$

$S(a,q) = \frac{1}{n_a} \sum_{k=1}^{n_a} |L_X(a,k)|^q$



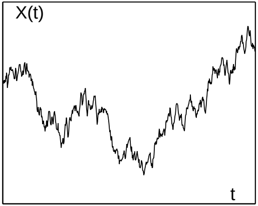
$\Longrightarrow$

$S(a,q) \simeq c_q a^{\zeta(q)}, \quad a \to 0$



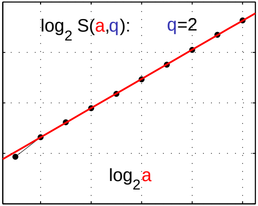
Multifractal Formalism (Analysis tool 3)

$X(t) \rightarrow d_X(a,t) \rightarrow L_X(a,t)$



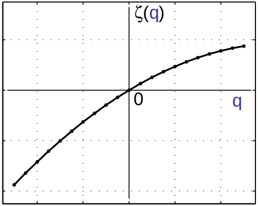
$\Downarrow$

$S(a,q) = \frac{1}{n_a} \sum_{k=1}^{n_a} |L_X(a,k)|^q$



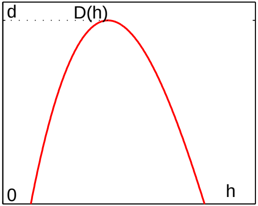
$\Longrightarrow$

$S(a,q) \simeq c_q a^{\zeta(q)}, \quad a \rightarrow 0$



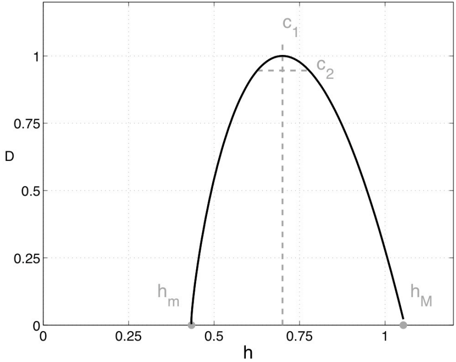
$\Uparrow$

$D(h) = \min_{q \neq 0} (d + qh - \zeta(q))$



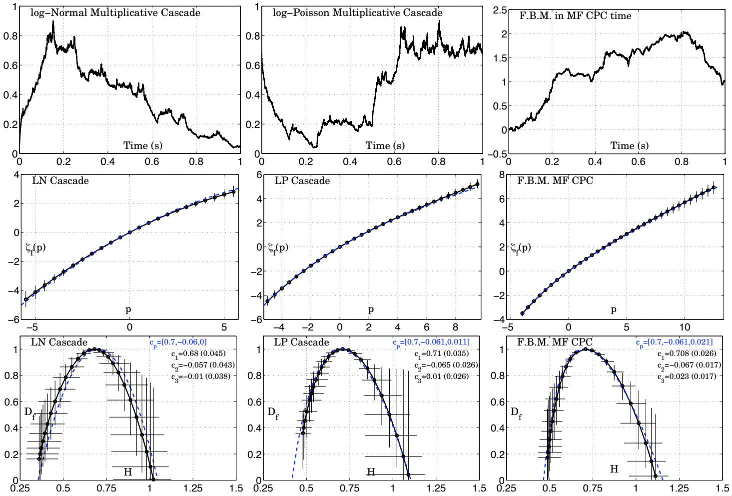
Multifractal Spectrum

- c_1: Location of max, $c_2 < 0$: Width,
- c_3: asymmetry (hard to estimate)
- h_{min} Minimun regularity, h_{max} Maximum regularity
- $D(h) \simeq 1 + \frac{c_2}{2}\left(\frac{h-c_1}{c_2}\right)^2 + \frac{c_3}{6}\left(\frac{h-c_1}{c_2}\right)^3 + \dots$



Multifractal processes (Model 3)

Wendt et al., 2007, Signal Proc. Mag.



Scale-Free toolbox

Wendt et al., 2007, Signal Proc. Mag., Wendt et al., 2009, Signal Processing

Scale-free and Multifractal WebSite

Toolbox

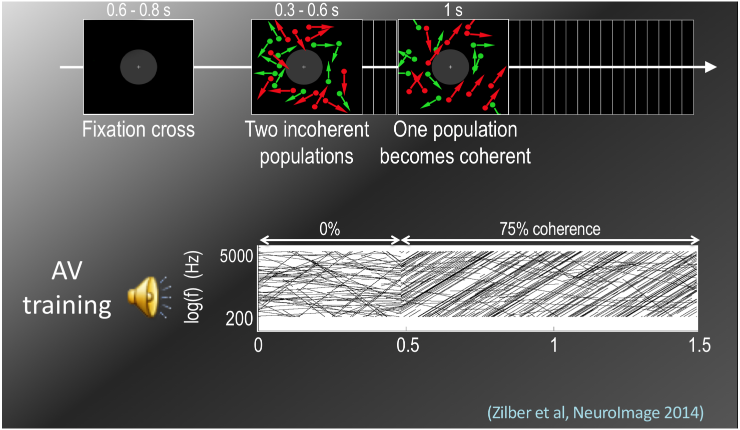
Tutorials

Multifractality in Brain ?

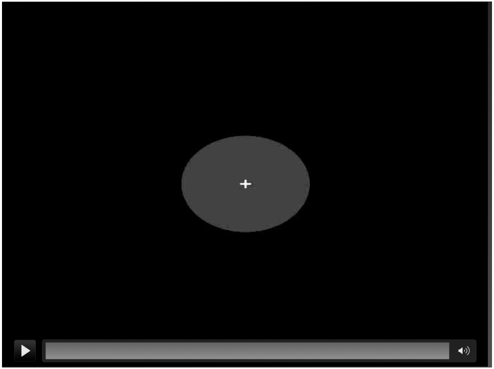
- Any Selfsimilarity/multifractality in brain ?
- Are they modulated by task ?
- Does multifractality relate to selfsimilarity ?
- What do they model in signals ?
- What do they model in brain activity ? Functional role ?

Outline

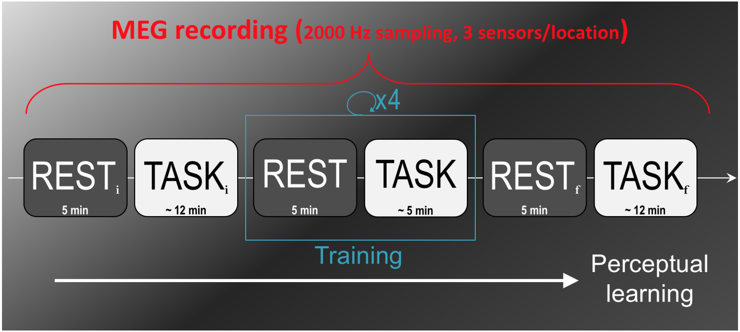
Experiment Design



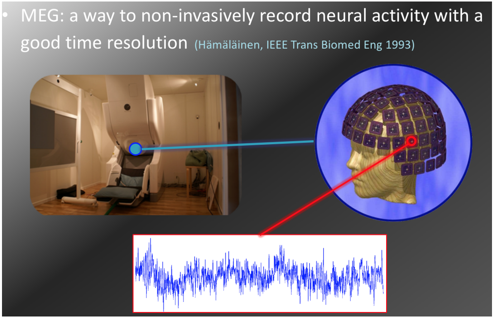
Experiment Design



Experiment Design



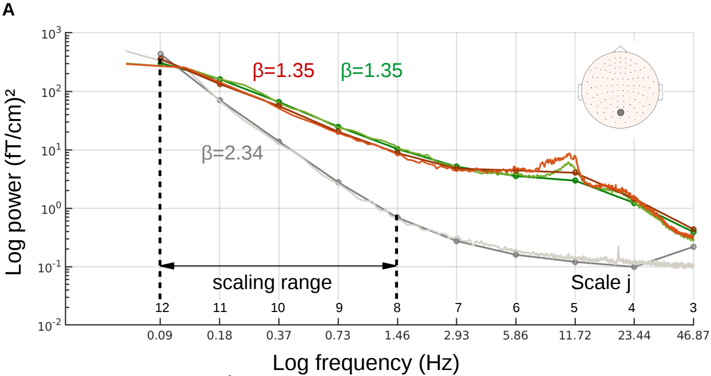
Data Collection (MEG)



- 24 (right-handed) participants, (10 women, age 22 ± 2)
- Sampling frequency: 400 Hz
- Regular MEG preprocessing (eye-blink, heart beat,...)
- MEG Source reconstruction (Destrieux atlas, 138 parcels)

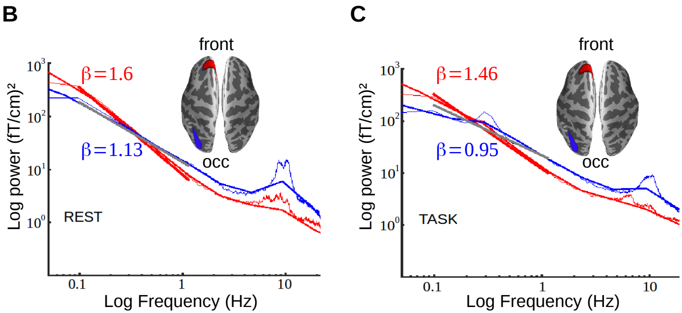
Outline

Scale Free is not spurious



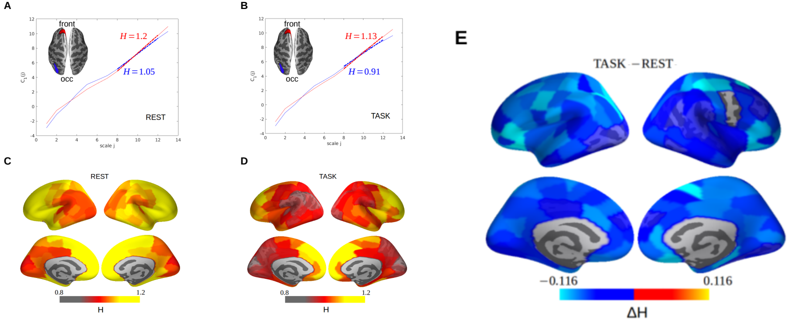
- Empty room recording:
  - Different power
  - Different scaling exponents
  - Different scaling range

Infraslow Scale Free dynamics



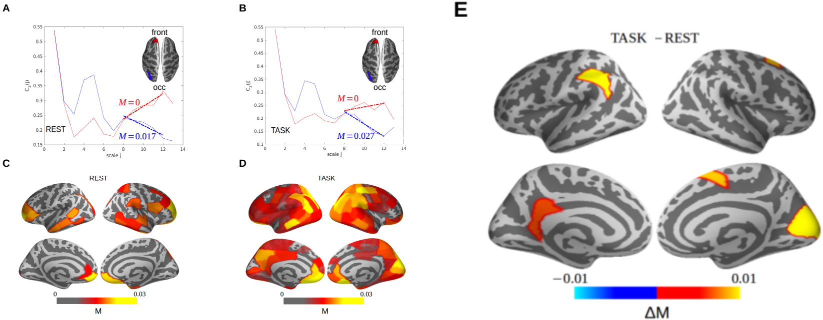
- Scale-free dynamics frequency range:
  - Octaves: $j_1 = 8 \leq j \leq j_2 = 12$
  - Frequencies: $f_{\text{min}} = 0.1 \leq f \leq= f_{\text{min}} = 1.5$ Hz
  - TimeScales: $a_{\text{min}} = 0.7 \leq f \leq= a_{\text{min}} = 10$ s
  - Both at rest and during task
  - Both for self-similarity and multifractality

Self-Sim.: Fronto-occipital gradient & task modulation



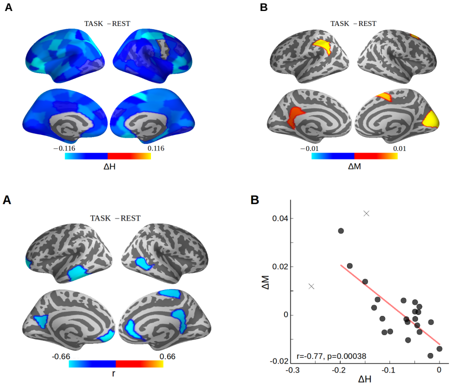
- Self-Similarity:
  - Rest: Fronto-occipital Self-Similarity gradient
  - Task vs Rest: Overall decrease in Self-Similarity
  - Task vs Rest: Overall increase of the fronto-occipital Self-Similarity gradient

Multifractality: task modulation



- Multifractality:
  - Rest: Weak ($M \simeq 0.01$) multifractality (mostly in Default Mode Network)
  - Task: localized increase of multifractality in regions engaged in the task (visual, parietal and motor cortices)

Covariation of Self-Similarity and Multifractality



- Covariation:
  - Anticorrelation: a decrease in H comes with an increase in M
  - Not theoretically mandatory thus a real data feature

Outline

Scale free dynamics in Brain

Modeling

Analysis

Multifractal

Data

Scale-free dynamics

Conclusions

Discussions and conclusions

La Rocca et al. , 2018

- Self-Similarity: Fronto occipital gradient
  - Larger H (slower dynamics) in high level regions
  - Lower H (faster dynamics) in sensory regions
  - overall decrease, Increased gradient, under task
- Multifractality: weak at rest (DMN)
  - Increase under task (localized to engaged regions)
  - burstiness, non-Gaussian, localized, beyond correlation
  - anticorrelated to Self-Similarity
- Interpretations:
  - ⇒ Rest: well globally structured, no burtiness
  - ⇒ Task: less global, more local, structures, beyond correlation
  - ⇒ Self-Similarity: Integration?
    Hierarchy of temporal dynamics? Neural excitability?
  - ⇒ Multifractality: Multiplexing?
- References:
  - patrice.abry@ens-lyon.fr ; perso.ens-lyon.fr/patrice.abry/
  - www.ens-lyon.fr/PHYSIQUE/Equipe3/MultiFracs/

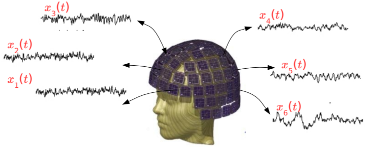
Limitations

Current MF analysis does well when data are:

- 1 - Univariate
- 2 - Homogeneous in time/space
- 3 - Isotropy
- 4 - Hölder only

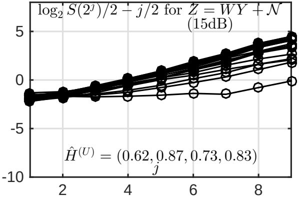
Limitation 1a: Univariate → Multivariate Self-Similarity

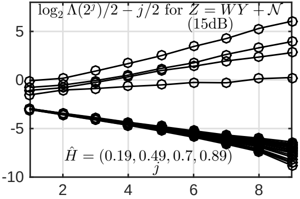
La Rocca et al. , 2018b



- Univariate analysis when many components recorded jointly ?
- Multivariate self-similarity analysis ?
  - Same H ? How many different H ? Large dimension ?

Abry, Didier, Hui, 2018, Abry, Didier, 201b8, Abry, Didier, 2018





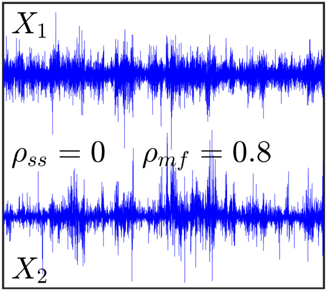
Limitation 1b: Univariate → Multivariate Multifractality

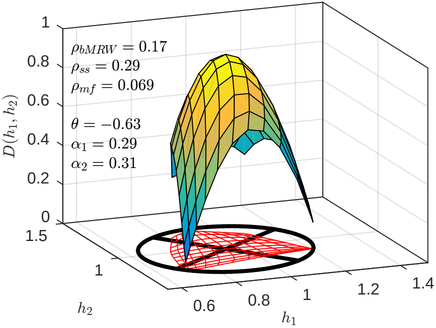
- Univariate analysis:
  even when many components recorded jointly.

Leonarduzzi et al. , 2018, Leonarduzzi et al. , 2017, Leonarduzzi et al. , 2016

- Multivariate analysis ?
  - Multivariate multifractal spectrum $D(h_1, \ldots, h_p)$?
  - Definition ? Estimation ? Meaning ? Use ?

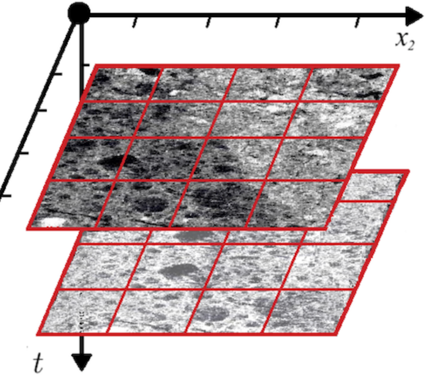
Jaffard et al. , 2019, Abry et al. , 2019, Jaffard et al. , 2018

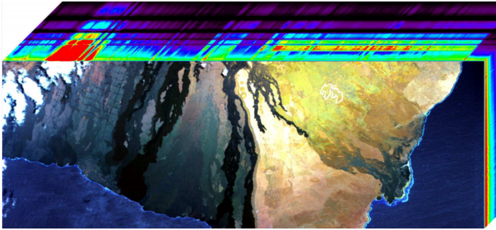




Limitation1c: Multivariate scale-free analysis ?

Combrexelle et al. 2015, IEEE TIP, Wendt et al., 2018, SIIMS,





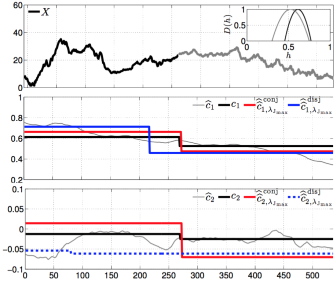
Limitation 2: Homogeneous → heterogeneous

Frecon et al. 2017, IEEE TSP

- Homogeneous (stationary)

  No Change along time

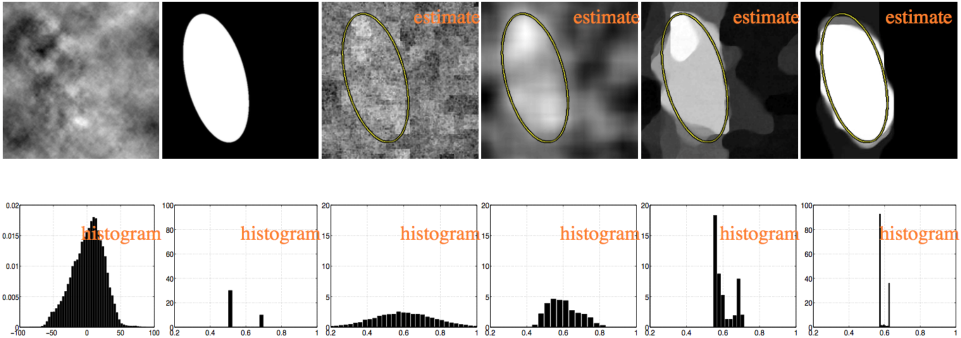
- Heterogeneous: need to Estimate jointly
  - change points and
  - MF properties in each region



Limitation2: non homogeneous texture

Pustelnik et al. 2016, IEEE

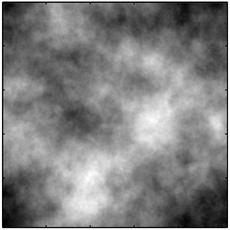
TCI



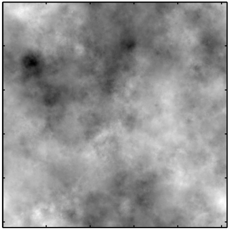
(a) Data $\underline{\underline{f}}$ (b) Original $\underline{\underline{h}}$ (c) $\widehat{\underline{\underline{h}}}$ (d) $\widehat{\underline{\underline{h}}}^{S}$ (e) $\widehat{\underline{\underline{h}}}^{TV}$ (f) $\widehat{\underline{\underline{h}}}^{TVW}$

Limitation3: Multifractality and Anisotropy ?

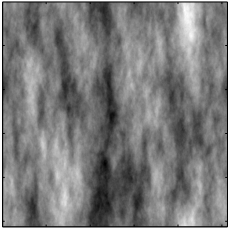
Isotropic Self-Similarity



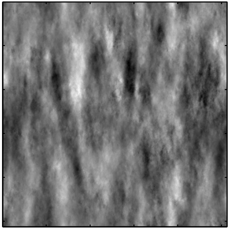
Isotropic Multifractality



Anisotropic Self-Similarity

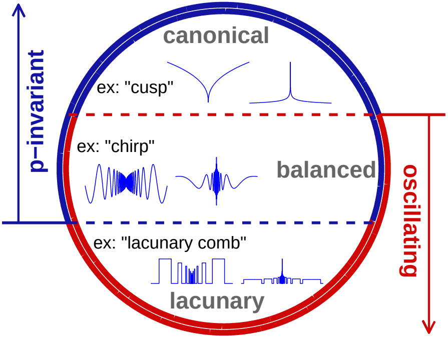


Anisotropic Multifractality



Limitation 4: Beyond Hölder

- Hölder like:
  - → $h(t) > 0, \forall t$,
  - → Cusp only
  - → Concave Spectrum
- Beyond Cusp ?



Leonarduzzi et al. , 2017, Leonarduzzi et al. , 2016

  - Regularity exponents 2.0:
    p-exponent, lacunarity, oscillation, cancellation exponents
  - Cross-frequency-coupling ?
- beyond Concave ? Leonarduzzi et al. , 2018

References - Multivariate SelfSimilarity

- Multivariate SelfSimilarity Analysis:

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Thank you ! Questions ?

Log-Cumulants

- For certain classes of processes :
  - $\mathbf{E} L_X(j,\cdot)^q = F_q|2^j|^{\zeta(q)}$

- 2^{nd} characteristic function $\ln L_X(j,\cdot)$:
  - $\ln \mathbf{E} e^{q \ln L_X(j,\cdot)} = \sum_p C_p^j \frac{q^p}{p!} = \ln F_q + \zeta(q) \ln 2^j$

    C_p^j: cumulant of order $p \geq 1$ de $\ln L_X(j,\cdot)$

- $\Rightarrow \forall p \geq 1 : \quad C_p^j = c_p^0 + c_p \ln 2^j$
  - $$\ln \mathbf{E} e^{q \ln L_X(j,\cdot)} = \underbrace{\sum_{p=1}^{\infty} c_p^0 \frac{q^p}{p!}}_{\ln F_q} + \underbrace{\sum_{p=1}^{\infty} c_p \frac{q^p}{p!}}_{\zeta(q)} \ln 2^j,$$
  - $\zeta(q) = \sum_{p=1}^{\infty} c_p \frac{q^p}{p!}$

Log-Cumulants

- For certain classes of processes :
  - $\mathbf{E} L_X(j,\cdot)^q = F_q |2^j|^{\zeta(q)}$
- 2^{nd} characteristic function $\ln L_X(j,\cdot)$:
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- For certain classes of processes :
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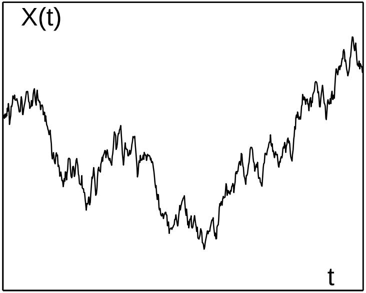
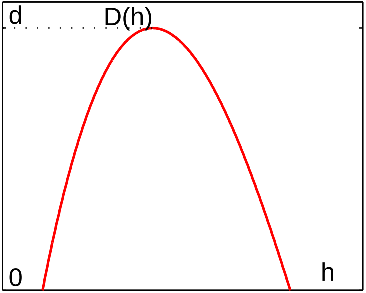
Multifractal Spectrum

- **Multifractal Spectre** $D(h)$:
  - Irregularity: Fluctuations of regularity $h(t)$
  - Set of points that share same regularity $\{t_i | h(t_i) = h\}$
  - Fractal (or Haussdorf) Dimension of each set:

$$D(h) = \dim_H\{t : \ h(t) = h\}$$





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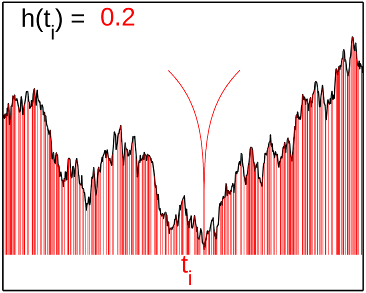

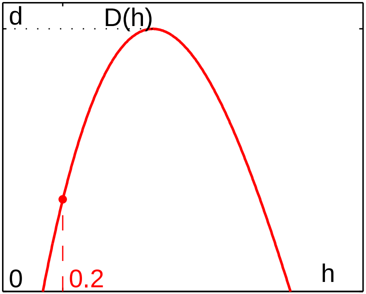

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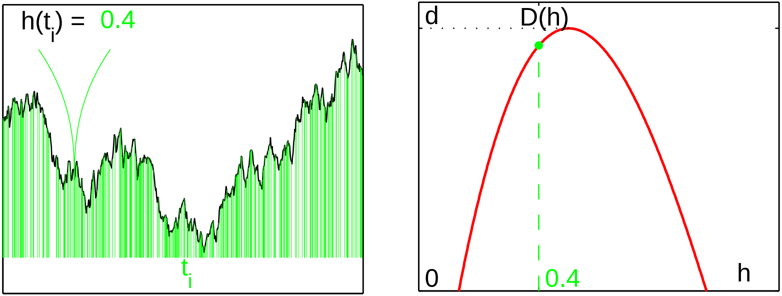

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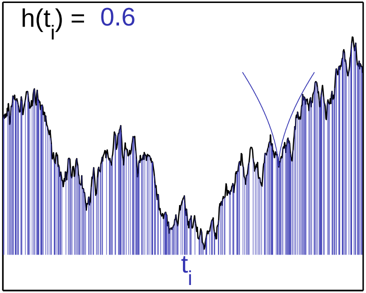

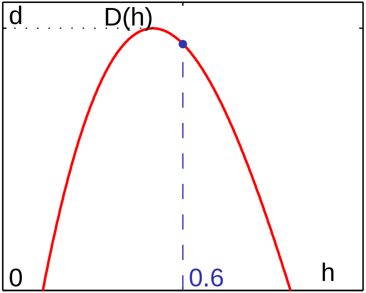

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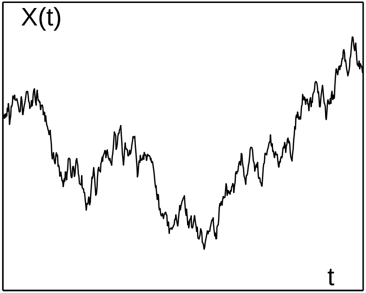
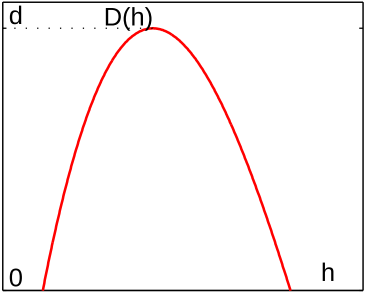
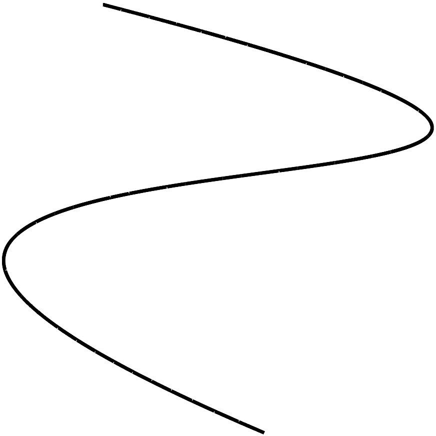
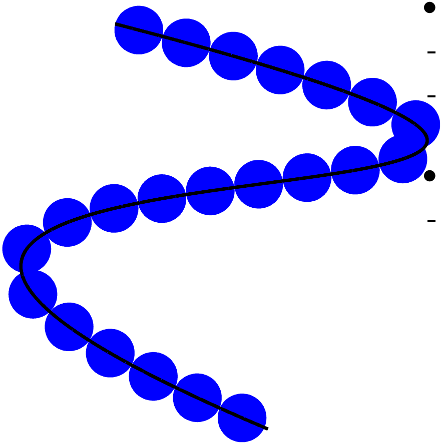




Dimension of a geometrical set

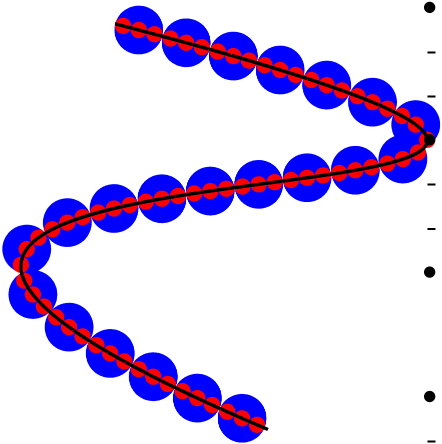


Euclidean dimension



- Let
  - a ($= 1$) be the analysis scale,
  - N denote the number of covering boxes with size a,
- Then
  - Length is : $L = N \cdot a$

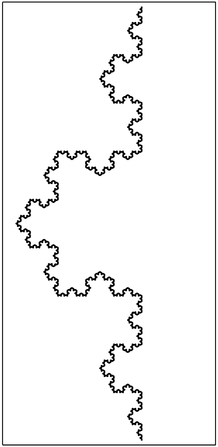
Euclidean dimension



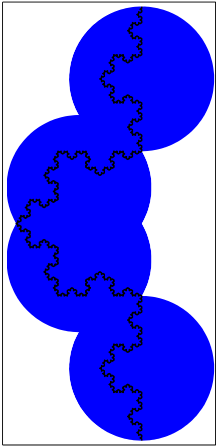
- Let
- a $(= 1)$,
- a $(= 1/3)$,
- hence,
- $N = \frac{a}{a} \cdot N \, (= 3 \cdot N)$,
- $L = N \cdot a = L = N \cdot a = L_0$,
- donc

  $L(a)$ does not depend on a nor on a !
- and
- $L(a) = N(a) \cdot a = L_0$,
- $\boxed{N(a) = L_0/a = L_0 \cdot a^{-1}}$.

Dimension of a geometrical set

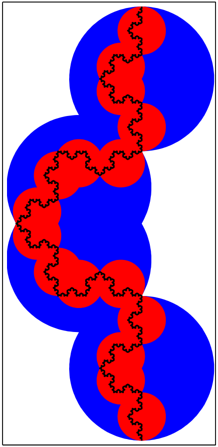


Fractal dimension



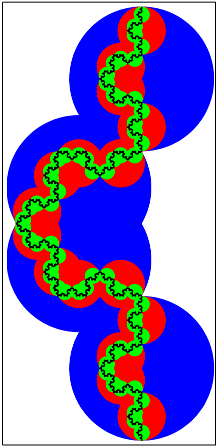
- Let
- a , be the analysis scale
- N denote the number of covering boxes with size a,
- Then
- Length is : $L = N \cdot a$
- Here,
- $a = 1/3$,
- $N = 4$,

Fractal dimension



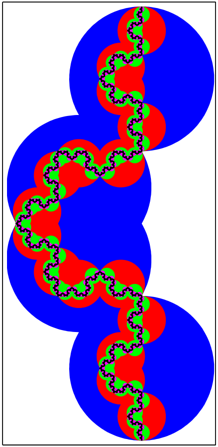
- Let
- $\color{blue}{a}$ $(= 1/3)$,
- $\color{red}{a}$ $(= 1/9)$,
- Then,
- $\color{blue}{N} = 4$,
- $\color{red}{N} = 16$,
- Hence
- $\color{blue}{L = N \cdot a} \neq \color{red}{L = N \cdot a}$! ,

Fractal dimension



- Let
- $a\,(= 1/3)$,
- $a\,(= 1/9)$,
- $a = 1/27$,
- Then,
- $N = 4$,
- $N = 16$,
- $N = 64$,
- donc
- $L = N \cdot a \neq L = N \cdot a \neq L = N \cdot a$!

Fractal dimension



- One shows:
- $a(n) = (1/3)^n$,
- $N(n) = 4^n$,
- hence
- $L(a) = N(a) \cdot a$,
- $L(a)$ does depend on a !
- with,
- $\boxed{N(a) = a^{-D}}$,
- $L(a) = L_0 \cdot a^{1-D}$,
- D : fractal dimension,
- $1 < D < 2$,
- non integer = Frac-.

Haussdorf Dimension

- Intuition:

  Fractal dimension,
  Non integer extension of the natural *Euclidean* dimension,
  $0 \leq D \leq d$.
  Cover a set A with balls of size ϵ, Count how many you need $N(\epsilon)$.
  Assume a power law behaviour $N(\epsilon) \sim \epsilon^{-D}$.
  Define $D = \lim_{\epsilon \to 0} - \log N(\epsilon) / \log \epsilon$.

- Definition:

  $A \in \mathcal{R}^d$,
  $\epsilon > 0, R$ ϵ-covering of A with a countable collection of bounded sets A_i, $|A_i| \leq \epsilon$,
  $\delta \in [0, d], M_\epsilon^\delta(A) = \inf_R \left(\sum_i |A_i|^\delta \right), M^\delta(A) = \lim_{\epsilon \to 0} M_\epsilon^\delta(A)$,
  D is such that $\delta > D, M^\delta(A) = 0, \delta < D, M^\delta(A) = \infty$.

Thermodynamic analogy (Parisi-Frisch, 85)

Thermodynamic	Multifractal
- $Z_\beta(U) = \sum_k e^{-\beta E_k}$,	- $S(a,q) = \sum_k \lvert T_X(a,k)\rvert^q$ $S(a,q) = \sum_k e^{q \log \lvert T_X(a,k)\rvert}$
$U = \langle E_k \rangle = \partial \log Z_\beta / \partial\beta$	- $\lvert T_X(a,k)\rvert = a^{h_k}$, - $S(a,q) = \sum_k e^{q h_k \log a}$
- β	- q
- $E_k = \epsilon_k \delta V$,	- $h_k \log a$, - $S(a,q) = a^{\zeta(q)}$,
- $F = -\ln Z_\beta$	- $\zeta(q) \log a = \log S(a,q)$,
- Entropy: $F = U - S/\beta$ (Legendre transform)	- Spectrum: $D(h) = qh - \zeta(q)$ (Legendre transform)

◂ to MF Form.

Rényi entropy
Strange attractors and chaotic systems (Kadanoff, 75)

- Rényi entropy: $Z_\alpha(a) = \sum_k P_k(a)^\alpha$,
- Rényi information: $I_\alpha(a) = \log Z_\alpha(a)/(1-\alpha)$,
- Generalized dimensions: $D_\alpha = \lim_{a\to 0} I_\alpha(a)/(-\log a)$,

$\Rightarrow \quad (1-\alpha)D_\alpha = \lim_{a\to 0} \log Z_\alpha(a)/\log a \equiv \zeta(\alpha)$!

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